$\cancel{4}$ 3-dimensional gauge theory and representation theory

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In Nov. 1996, Witten gave a series of lectures at
Newton Institute, Cambridge.

He explained 3 topics:

1. 3-manifolds invariants via hyper-Kaehler manifolds
   (Rozansky-Witten)

2. Coulomb branch for 3d N=4 SUSY gauge theory
   (Seiberg-Witten for SU(2) and others)

3. 3d mirror symmetry (Intriligator-Seiberg) and
   its description by branes (Hanany-Witten)

They were instructive lectures, and I learned quite lots.

But I could not produce my own work out of them at that time.

Today : My response to lectures, 18 years later !

3d N=4 SUSY gauge theory (topologically twisted version)

data

- G : a compact Lie group
- V : a quaternionic representation of G $\quad (G \times Sp(1))$, $Sp(1) \cong Spin(3)$

(- P : a principal G-bundle over M^3)

fields

- A : G-connection + etc (vector multiplet)
- Φ : spinor for $(P \times P_{spin}) \times_{G \times Sp(1)} V$ + etc (hypermultiplet)

$\rightsquigarrow$ a hyper-Kaehler manifold $\mathcal{M}_C$: Coulomb branch

$\mathcal{M}_C$ is important to understand the gauge theoretic 3d invariants, defined by counting solutions of PDE of fields.

e.g. Casson – Walker invariant $(G = SU(2), V = 0)$
3d monopole invariant $(G = U(1), V = \mathbb{H})$
$=$ Rozansky-Witten invariant of $\mathcal{M}_C$

Seiberg-Witten (1996)

$G = SU(2)$, $V = 0$ $\Rightarrow$ $\mathcal{M}_C =$ Atiyah-Hitchin manifold

$G = U(1)$, $V = \mathbb{H}$ $\Rightarrow$ $\mathcal{M}_C =$ Taub-NUT space

$\mathcal{M}_C$ is based on quantum field theory, which lacks mathematical foundation. For mathematicians, claims sound as

(something, not defined) = AH, or TN space.

Thus mathematicians cannot understand these statements.

Later $\mathcal{M}_C$ were computed in many examples in physics literature, interesting hyper-Kaehler manifolds, such as instanton moduli spaces of exceptional groups, appeared.

Examples.

1. Abelian gauge theory

$$G = U(1)^n \curvearrowright \mathbb{H}^d$$

$$\Longrightarrow \quad 0 \to \mathbb{Z}^n = \pi_1(G) \longrightarrow \mathbb{Z}^d = \pi_1(U(1)^d) \longrightarrow \mathbb{Z}^{d-n} \longrightarrow 0$$

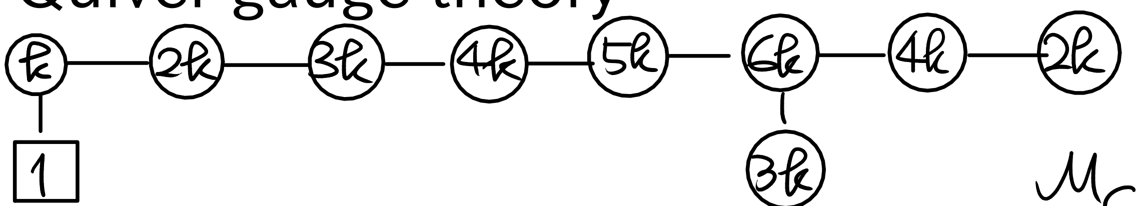
Apply $\mathrm{Hom}(\cdot, U(1))$:

$$1 \to U(1)^{d-n} \longrightarrow U(1)^d \longrightarrow G^\vee \text{ (dual torus)} \longrightarrow 1$$

Then $\quad \mathcal{M}_C = (\mathbb{H}^d)^* /\!/\!/ U(1)^{d-n} \quad$ (hyperkähler quotient)

cf. $\mathcal{M}_H = \mathbb{H}^d /\!/\!/ G$

2. Quiver gauge theory $\qquad (k \in \mathbb{Z}_{>0})$



$\mathcal{M}_C$ = moduli space of k E_8-instantons on $\mathbb{R}^4$

It was a similar to Seiberg-Witten curves for 4d N=2 SUSY gauge theory. Analogue of $\mathcal{M}_C$ was determined as a family of elliptic curves (Seiberg-Witten curves).

A mathematical definition of (analogue of) $\mathcal{M}_C$ was given by Nekrasov much later in 2002 by Ω-background.
And the theory has been developed further after that.

I learned from this lesson:

> Writing down a mathematically rigorous definition
> is a non-trivial, often difficult, but challenging problem.

Today : I propose a mathematically rigorous definition of $\mathcal{M}_C$,

at least as a complex manifold (for good or ugly theories).
(work in progress with Braverman+Finkelberg)

$\Sigma = \mathbb{P}^1$
$\mathcal{R}$ = moduli stack of pairs

- $\mathcal{P}$: a holomorphic $G_{\mathbb{C}}$ -bundles over Σ
- Φ : a holomorphic section of $(\mathcal{P} \times_{G_{\mathbb{C}}} V) \otimes \mathcal{O}_{\Sigma}(-1)$ such that

$\mu_{\mathbb{C}}(\Phi) = 0$

$\mathcal{R}$ is a critical point of a complex Chern-Simons type functional,

hence has a sheaf φ_{CS} of the vanishing cycle.

$$\boxed{\mathbb{C}[\mathcal{M}_C] = \text{(dual of) } H_c^*(\mathcal{R}, \varphi_{CS})}$$

$U(1) \subset SU(2)$

cohomology degree

(Multiplication will be explained later.)

Reasons why this definition should give the Coulomb branch $\mathcal{M}_C$

1. It reproduces the monopole formula of Cremonesi-Hanany-Zaffaroni, found recently (at least if $V = \underset{\circlearrowright G}{V'} \oplus \underset{\circlearrowright}{(V')^*}$).
2. When V = 0 (which violates good or ugly assumption), Bezrukavnikov-Finkelberg(-Mirkovic) identified $\mathcal{M}_C$ with a certain symplectic quotient, which is moduli space of charge k SU(2)-monopoles (for SU(k)) (Bielawski), as predicted by Seiberg-Witten (k=2) and Chalmers-Hanany.
3. True for abelian
4. A natural quantization

1. $G_{\mathbb{C}}$ - bundles over $\mathbb{P}^1$ are reduced to $T_{\mathbb{C}}$-bundles.
(Vector bundles over $\mathbb{P}^1$ are direct sum of line bundles.)

This induces a stratification of $\mathcal{R}$, which is *perfect*.

Therefore the dimension of the cohomology is the sum of dimension of cohomology of strata.

$$P_t(H_c^*(\mathcal{R}, \varphi_{cs})) = \sum_{\lambda \in \text{coweight of } G / \text{Weyl group}} \frac{1}{P_t(H^*_{\mathrm{Stab}(\lambda)}(pt))} \times t^{2\Delta(\lambda)}$$

$$\Delta(\lambda) = -\sum_{\alpha \in \Delta^+} |\langle \alpha, \lambda \rangle| + \frac{1}{2} \sum_{\substack{b:\ \mathbb{H}\text{-base} \\ \text{of } V}} |\langle w + b, \lambda \rangle|$$

Product

Taking the Poincare dual, we switch from bundles over $\mathbb{P}^1$ to

affine Grassmannian $Gr_{G_{\mathbb{C}}}$

= bundles over $\mathbb{P}^1$ + trivialization away 0

$Gr_{G_{\mathbb{C}}} \sim \Omega G$: based loops group

$\Longrightarrow$ Pontryagin product $\quad H_*(\Omega G) \otimes H_*(\Omega G) \longrightarrow H_*(\Omega G)$